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## DISCRETE AMBIGUITIES IN PHASE-SHIFT ANALYSIS

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**Abstract:** In two practical examples ( $\alpha$ - $^3\text{He}$  and  $\alpha$ - $\alpha$  scattering) we investigate to what extent the elastic amplitude above the first inelastic threshold, determined from phase-shift analysis, is subject to ambiguity. We find that it is extremely difficult to determine the correct physical amplitude uniquely.

### 1. Introduction

An old and intriguing problem in modern physics is the relationship between quantities which are measured in a scattering experiment, and the interaction between the particles taking part in the scattering process. This problem consists of two parts: the first is the determination of a scattering amplitude from the experimental data, and the second is the so-called inverse scattering problem, where one tries to find the interaction from a given amplitude. It is a well-known fact that even if the amplitude is known at all energies between threshold and infinity, the interaction cannot be uniquely determined<sup>1)</sup>. The problem of non-uniqueness in the construction of unitary scattering amplitudes from experimental data has only recently been investigated more closely.

In elastic two-body collisions one can measure the differential cross section and, if particles with spin are present, the polarization and spin-correlation parameters. At energies above the first threshold for inelastic collisions one can also measure the corresponding quantities for the inelastic processes. Another measurable quantity in this energy region is the total cross section. We shall in the following consider only the spinless case. The formalism can be easily extended to spin-0–spin- $\frac{1}{2}$  scattering [refs. 2, 3]).

In single-energy phase-shift analysis one tries to find an amplitude  $F$  in the physical region  $-1 \leq \cos\theta \leq +1$ , where  $\theta$  is the scattering angle in the centre-of-mass system (c.m.s.), when the differential cross section, equal to  $|F|^2$  except for kinematical factors, is given. It is convenient to parametrize  $F(x = \cos\theta)$  with the phase shifts  $\delta_l$  and the elasticities  $\eta_l$  of its partial waves  $f_l$ :

$$f_l = \frac{1}{2} \int_{-1}^{+1} dx P_l(x) F(x) = \frac{1}{2i} (\eta_l e^{2i\delta_l} - 1), \quad (1.1)$$

since unitarity is then satisfied if one restricts  $\eta_l$  to be between 0 and 1. This constraint

and the knowledge of  $|F|^2$  are not sufficient to determine the amplitude  $F$  uniquely, and in the following we shall concern ourselves with the construction of ambiguous phase-shift solutions. The important feature of all ambiguities that we shall consider is that they are in no way dependent upon experimental errors, i.e. they exist even if the differential cross section is known with infinite accuracy at all angles.

We shall restrict ourselves to amplitudes which have a finite number  $L+1$  of partial waves, and we shall consider transformations that leave  $|F|$  and  $L$  unaltered. We exclude then most of the so-called continuum ambiguity<sup>4)</sup>, which is associated with transformations that change  $F$  continuously from its original value, while keeping  $|F|$  constant. The amplitudes associated with the continuum ambiguity have in general an infinity of exponentially decreasing partial waves. The continuum ambiguity, which can exist only above the first inelastic threshold, is considered in detail elsewhere<sup>5)</sup>. For elastic scattering below the first inelastic threshold (all  $\eta_l = 1$  in (1.1)) examples of non-trivial phase-shift ambiguities have been constructed<sup>6)</sup>. In these examples (for  $L = 2$  and  $L = 3$ ) the amplitudes (and therefore  $|F|$ ) are almost completely determined by the requirements  $|F| = |F'|$  and  $\eta_l = \eta'_l = 1$  for all  $l$ , so that the differential cross sections for which these ambiguities occur are necessarily rather specific. It has been shown recently that it is possible to construct such ambiguities for other cross sections<sup>7)</sup>, but the amplitudes will then have in general an infinite number of partial waves. In the region above the first inelastic threshold the inelasticities  $\eta_l$  are in principle unknown, and one does not have to require that  $\eta_l$  be equal to  $\eta'_l$ . This freedom makes it possible to construct ambiguities in physical cases in the inelastic domain.

As an example that is free of spin and isospin complications we have considered  $\alpha$ - $\alpha$  scattering. In addition we have explored cross-section data for  $\alpha$ - $^3\text{He}$  scattering under the simplifying assumption that  $^3\text{He}$  can be considered spinless. In both examples we have ignored the effect of the Coulomb interaction between the particles by working only with purely nuclear amplitudes. Such a procedure would be justified at high energies where Coulomb effects are small, or in cases where few forward direction data are available. For the processes and energies considered in this paper this is hardly the case, and our results be interpreted mainly as an illustration of what may happen when the Coulomb parameter  $Z_1 Z_2 e^2 / \hbar v$  is zero or small. One can show by analyticity arguments that if, for scattering of charged particles, the modulus of the full amplitude (i.e. Coulomb plus nuclear) is known exactly over an interval of scattering angles there will no longer be any ambiguities. In practice it is hard to exploit such analyticity properties, since this would require very accurate near-forward direction measurements of the differential cross section, and we therefore still expect to have ambiguities, although they will be of a more approximate nature. In fact, we have observed that if we used our "purely nuclear ambiguities" as a starting point in a  $\chi^2$  fit to the experimental data, it was always possible to find a nearby minimum of  $\chi^2$ . In future work we hope to consider this Coulomb problem in more detail.

In the next section we introduce the notation. The method of constructing ambi-

guities is discussed, and in particular we consider also the case where the total cross section is measured. In sect. 3 we discuss the properties of the essential ingredient of our analysis, the complex zeros  $z_i$  of  $F(x)$ . The results for  $\alpha$ - $\alpha$  and  $\alpha$ - $^3\text{He}$  scattering are given in sect. 4. Finally, in sect. 5 we list some conclusions.

## 2. Discrete ambiguities

We limit our treatment to spinless, elastic scattering at fixed energy. The differential cross section is expressed in terms of a complex scattering amplitude  $F(x)$  by:

$$q^2 \frac{d\sigma}{d\Omega} \equiv \sigma(x) = |F(x)|^2. \quad (2.1)$$

Here  $q$  is the c.m.s. momentum, and we shall henceforth suppress it. The requirement that  $F(x)$  be unitary is most easily expressed in terms of the partial waves (1.1) by the inequality

$$\text{Im } f_l \geq |f_l|^2, \quad (2.2)$$

which reduces to an equality below the first inelastic threshold. We express  $F(x)$  in terms of a finite number of parameters by setting all partial waves with  $l$  greater than a certain  $L$  equal to zero. Then  $F(x)$  becomes a polynomial of degree  $L$  in  $x$ :

$$F(x) = \sum_{i=0}^L (2i+1)P_i(x) \frac{\eta_i e^{2i\delta_i} - 1}{2i}. \quad (2.3)$$

Then the  $2L+2$  parameters  $\eta_i, \delta_i$  are varied to minimize:

$$\chi^2 = \sum_{n=1}^N \left| \frac{\sigma(x_n) - |F(x_n)|^2}{\Delta\sigma(x_n)} \right|^2, \quad (2.4)$$

where  $\sigma(x_n)$  is the measured value of the differential cross section at  $x_n$ , and  $\Delta\sigma(x_n)$  its error. The quantity  $\chi^2$  is for a given  $L$  a function of the parameters  $\eta_i, \delta_i$ . One of the problems of a phase-shift analyst is that for a given  $L$  there may be many minima of  $\chi^2$  with acceptable  $\chi^2$  values, and he has to find a sensible way to select a "best" solution among the set of solutions he has found. We wish to point out in this paper that a  $\chi^2$  criterion alone cannot suffice to select a best solution. To each solution of the minimization of (2.4) there correspond in general – above the first inelastic threshold – other solutions that are equivalent in the sense of a  $\chi^2$  test. It should be stressed that this ambiguity cannot be resolved by more precise measurements or by measurements at more angles. In fact, even if the differential cross section is known at all angles with infinite accuracy, this ambiguity is still present.

Since  $F(x)$  is chosen to be a polynomial of degree  $L$  in  $x$  we can also express it in terms of its  $L$  complex zeros  $z_n$  and one phase factor  $^2$ ):

$$F(x) = F(1) \prod_{n=1}^L \frac{x - z_n}{1 - z_n}. \quad (2.5)$$

We now consider the transformations that leave  $|F(x)|$  (and therefore the differential cross section) and the number of partial waves, unchanged.

(A) One or more of the complex zeros  $z_i$  are replaced by their complex conjugates  $z_i^*$ . This leads to a phase-shift ambiguity only if the new amplitude  $F'(x)$  satisfies the unitary constraints (2.2).

(B) The transformation  $F \rightarrow F' = e^{i\phi} F$  where  $\phi$  is a real  $z$ -independent constant. This implies that all partial waves are multiplied by the same phase factor:  $f_l \rightarrow f'_l = e^{i\phi} f_l$ . The constant  $\phi$  is constrained by the requirement that all partial waves lie inside the unitarity circle.

(C) Combinations of the transformations A and B. It should be understood that in the case of combinations of A and B we need not require that A and B separately produce unitary amplitudes. Combinations of A and B are particularly important for those transformations A that do not give a unitary amplitude, since it may be possible to restore unitarity by a rotation B. An example of a C-transformation that always preserves unitarity is the sign transformation  $\delta_l \rightarrow -\delta_l$ , all  $l$ , produced by conjugating all zeros and multiplying by  $e^{i\phi}$ , where  $\phi$  is the angle between  $F(1)$  and  $-F^*(1)$ .

So far we have not considered the total cross section. The optical theorem relates the total cross section to the forward amplitude:

$$\sigma_{\text{tot}} = \frac{4\pi}{q^2} \text{Im } F(1). \quad (2.6)$$

It is clear that transformations of type A leave the total cross section (and also the reaction cross section) unchanged. Transformations B and C will generally change the total cross section. One may allow these changes if there is no measurement of the total cross section, as is the case in the examples we shall consider in sect. 4. If unitarity allows rotations of  $F(1)$  between the angles  $\phi_{\min}$  and  $\phi_{\max}$ , so that  $\phi_{\min} \leq \arg F(1) \leq \phi_{\max}$ , where  $0 \leq \phi_{\min} \leq \phi_{\max} \leq \pi$ , then  $\sigma_{\text{tot}}$  is bounded below by

$$\frac{4\pi}{q^2} |F(1)| \min \{ \sin \phi_{\min}, \sin \phi_{\max} \}, \quad (2.7)$$

and above by:

$$\frac{4\pi}{q^2} |F(1)| \max \{ \sin \phi; \phi_{\min} \leq \phi \leq \phi_{\max} \}. \quad (2.8)$$

The upper bound in (2.8) is equal to the maximum possible value  $(4\pi/q^2)|F(1)|$  if  $\phi_{\min} \leq \frac{1}{2}\pi \leq \phi_{\max}$ .

We shall now summarize the ambiguity structure for a fixed number of partial waves. First we consider the case where  $\sigma_{\text{tot}}$ , and therefore  $\text{Im} F(1)$ , is kept at a fixed value. If our scattering amplitude has  $L+1$  partial waves, and therefore  $L$  complex zeros  $z_i$ , there will be  $2^L$  possible ways to conjugate some or all of the zeros  $z_i$  (if none of the zeros  $z_i$  happens to be real). The rotation that changes  $F(1)$  into  $-F^*(1)$  (we shall call this transformation R) also leaves  $\sigma_{\text{tot}}$  unchanged, bringing the total

number of ambiguities up to  $2^{L+1}$ . These possibilities will have to be checked for unitarity.

If we decide to leave  $\sigma_{\text{tot}}$  free, we have the same set of  $2^{L+1}$  possible ambiguities, but now each one of these can also be rotated over an angle  $\phi$ . We shall call these transformations  $B(\phi)$ . We wish to stress once more that with the rotations  $B(\phi)$  one cannot merely extend those ambiguities from the set A and RA that satisfy the unitarity constraints, but it is also possible in some cases to rotate the partial waves of A-transformed amplitudes that violate unitarity back inside the unitarity circle.

### 3. Zeros of the scattering amplitude

In sect. 2 we have discussed a method for constructing all scattering amplitudes with a given modulus and a given number of partial waves  $L+1$ . The essential quantities in this method are the complex zeros of the amplitude. In this section we shall give a very qualitative discussion of the connection between the positions and trajectories of the zeros, and the related ambiguity structure. A more detailed account of the properties of zeros of scattering amplitudes was given by Barrelet in a recent paper <sup>3)</sup>.

If we consider the energy dependence of the zeros it is, in most cases, possible to distinguish between two groups of zeros: a set of zeros that are close to the physical region and relatively stable as the energy changes, and a set of zeros that are further away from the physical region and rather unstable. The first set of zeros reflect directly the structure of the differential cross section, as their positions usually correspond to the positions of minima of  $\sigma(x)$ . The second set of zeros is of a different nature. They were called "statistical" zeros by Barrelet <sup>3)</sup>, and they strongly depend on experimental errors, the number of terms in the polynomial expansion, and so on.

In the some cases the effect on the amplitude in the physical region of a transformation A is very small; for instance, if only zeros very far from the physical region are conjugated and also if only zeros with very small imaginary parts are involved. In these cases one may hope to find new amplitudes satisfying the unitarity constraints. There are other configurations of two or more zeros that are likely to give rise to ambiguities. One may hope to resolve the ambiguities by considering the amplitudes at nearby energies and by demanding that the positions of the zeros are smooth functions of the energy. In fig. 1 we show some examples of trajectories of zeros that satisfy this smoothness condition, but still allow an ambiguity. A choice can be made between two alternative trajectories if one of them leads to a violation of unitarity upon further energy continuation. Some of the examples in fig. 1 were indeed encountered in our analysis of the ambiguities in  $\alpha$ - $\alpha$  scattering (sect. 4).

The position of the complex zeros as a function of the energy contain information about the dynamics of the scattering process. In particular, if we construct the function  $dz_k(E)/dE$  from the known position of the zero  $z_k(E)$  we find, in the case of  $\alpha$ - $\alpha$  scattering, pronounced structure in the neighbourhood of resonances. In fact, it

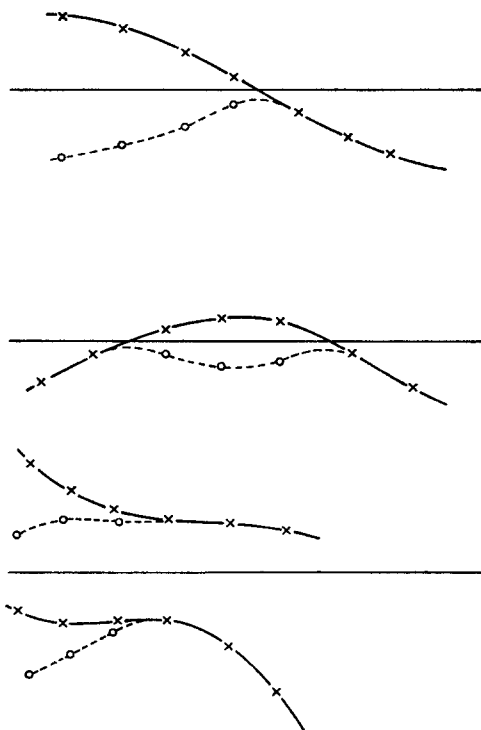


Fig. 1. Examples of trajectories of zeros of the scattering amplitude, which may give rise to ambiguity. Original (full line) and alternative (dotted line) paths are shown.

appears that if the  $l$ th partial wave resonates, the stable zeros correspond to zeros of  $l$ th Legendre polynomial, and at the resonance energy the derivative  $dz_k(E)/dE$  for these zeros shows a clear dip. In the case of  $\alpha$ - $\alpha$  scattering this structure is complicated by resonance overlap.

If the scattering amplitude is assumed to be a polynomial, the information contained in the measurements of the differential cross section is essentially the positions of the zeros of this polynomial (except for the conjugation transformations). It may therefore be advantageous to use these zeros directly as parameters in a fit to the experimental data, rather than the partial waves. Some of the advantages are obvious. For a polynomial of degree  $L$  we have  $L+1$  partial waves, and therefore  $2L+2$  parameters in a partial-wave fit, but we have  $L$  zeros and one real constant (if the phase of the forward amplitude is left undetermined), and therefore  $2L+1$  parameters in a zero fit. If a normalization parameter is required for the experimental data it can easily be absorbed into the real constant in the case of zero fitting. For each minimum of  $\chi^2$ , i.e. a  $\chi^2$  with zeros as parameters, we can easily construct all ambiguities with the method of sect. 2, and select the unitary amplitudes. If more than one unitary amplitude can be found (and if the total cross-section measurement is not available this

will certainly be the case), then all of these ambiguities correspond to minima of  $\chi^2_{\text{p.w.}}$ , the  $\chi^2$  with partial waves as parameters, and if a  $\chi^2$  search with partial waves is to be complete all of them will have to be found. This will in general be a more difficult and time-consuming task than the selection of unitary amplitudes from the fitted zeros of the amplitude.

Fitting with zeros as parameters has some clear advantages over the conventional method of fitting with partial waves, but we must realize that with the zero fits unitarity constraints are no longer satisfied by construction. In the cases where information about elasticities is available – when for instance some (or all) of the partial waves have to be elastic – the conventional method of partial-wave fits is to be preferred.

#### 4. Results

In this section we shall apply the methods of sect. 2 to two elastic scattering processes,

(a)  $^3\text{He} + ^4\text{He} \rightarrow ^3\text{He} + ^4\text{He}$ ,

(b)  $^4\text{He} + ^4\text{He} \rightarrow ^4\text{He} + ^4\text{He}$ .

We have used the results of existing phase-shift analyses as starting points in our investigation, and calculated all possible ambiguities in the low energy region above the first inelastic threshold.

TABLE 1

Values of the phase-shift parameters of Tang and Brown for elastic scattering  $^4\text{He} + ^3\text{He}$  at 44.5 MeV c.m.s.

$l$	$\delta_l$ (rad)	$\eta_l$
0	3.06480	0.36466
1	3.25853	0.56413
2	2.44695	0.33186
3	2.52200	0.52793
4	1.36136	0.23571
5	1.38579	0.44493
6	0.20595	0.45594
7	0.32463	0.68115
8	0.02618	0.86363
9	0.07156	0.93910
10	-0.00349	0.97586
11	0.01571	0.98958

For process (a) we used the results of a recent phase-shift analysis by Fetscher *et al.* <sup>8,9)</sup> and a model calculation by Tang and Brown <sup>10)</sup>. In both cases the experimental differential cross section was analyzed neglecting spin dependence. For this process this approach is used in the low energy region by most authors. The few measurements <sup>9)</sup> of the polarization do not indicate that the effect of the spin of  $^3\text{He}$  is significant. Tang and Brown used the resonating group method in their calculation. The existence of open reaction channels is taken into account by the introduction of a



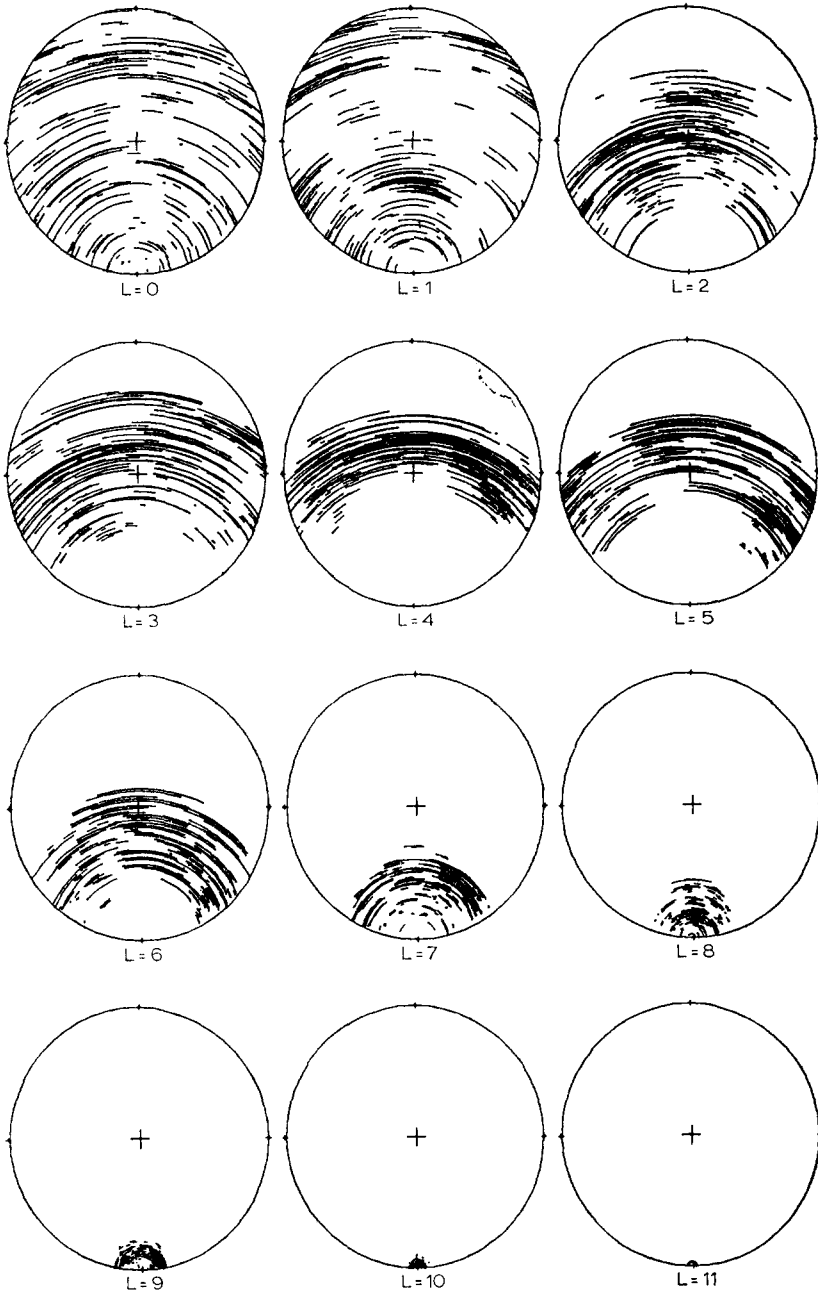


Fig. 2. Argand plots for ambiguous amplitudes corresponding to the differential cross section of Tang and Brown at 44.5 MeV c.m.s., with  $\sigma_{\text{tot}}$  free.

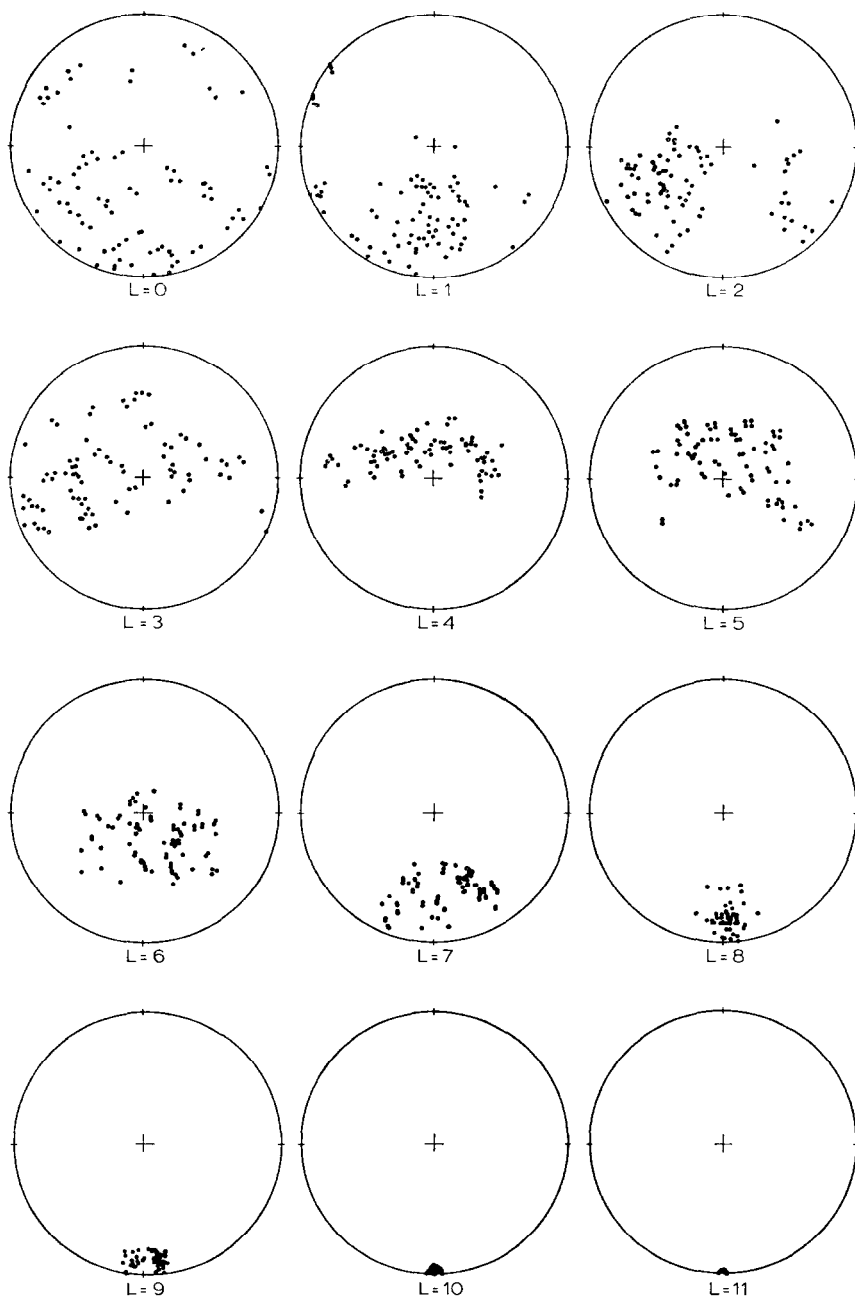


Fig. 3. Argand plots for ambiguous amplitudes corresponding to the differential cross section of Tang and Brown at 44.5 MeV c.m.s., with  $\sigma_{\text{tot}}$  kept fixed.

phenomenological local imaginary potential in the usual resonating group formalism of the scattering problem. By adapting the shape and strength parameters of this potential they find that a fairly good fit to the experimental differential cross section at 44.5 MeV(c.m.s.) can be obtained. The first twelve partial waves of their amplitude are given in table 1.

In fig. 2 ambiguous scattering amplitudes corresponding to the transformations A, B and C of sect. 2 are displayed in twelve Argand diagrams ( $\text{Im } f_l$  plotted against  $\text{Re } f_l$ ) for  $l = 0, 1, 2, \dots, 11$ . For each point on an arc in one of the Argand plots, there is a corresponding point in each of the other diagrams, such that the full amplitude reproduces exactly the differential cross section from Tang's amplitude. In fig. 2 we have not included the sign ambiguities  $\delta_l \rightarrow -\delta_l$ , all  $l$ .

In fig. 3 we display all ambiguities of type A. These amplitudes then have the same value for  $\text{Im } F(1)$  as Tang's amplitude, and therefore correspond to the same total cross section. The continuous curves in fig. 2 now reduce to points, since arbitrary phase changes  $B(\phi)$  are no longer possible. There are more curves in fig. 2 than points in fig. 3 since transformations of the type BA sometimes lead to unitary amplitudes, even though the transformation A does not. However, it is possible to find cases where more points than curves appear, because partial waves obtained through the transformations A and RA are in general on the same curve, but do give rise to two discrete points (unless  $\text{Re } F(1) = 0$  when the two points are the same). Of course all of the points in fig. 3 also appear in fig. 2. Note that the decrease of the partial waves with increasing  $l$  is not much affected by the transformations. In fact, one easily shows that  $|F_L|$  remains unchanged by any transformation, A, B or C.

TABLE 2  
Number of ambiguities as a function of  $L$

$L$	Upper bound $2^L$	No. of arcs with unitary solutions with $\sigma_{\text{tot}}$ free	No. of points representing unitary solutions with $\sigma_{\text{tot}}$ fixed
11	2048	184	83
10	1024	83	54
9	512	103	66
8	256	60	53

TABLE 3  
Number of ambiguities for the Fetscher amplitudes as a function of the energy

$E_{\text{c.m.s.}}$ (MeV)	$\sigma_{\text{tot}}$ free	$\sigma_{\text{tot}}$ fixed
28	13	8
32	21	16
37	23	22
40.5	20	16
44	13	5

TABLE 4

The phase shifts  $\delta_l$  and elasticities  $\eta_l$  for the Fetscher amplitudes at five energies, and for the ambiguity corresponding to the conjugation of one of the zeros of the original amplitude

$E_{c.m.s.}$	28.0		32.0		37.0		40.5		44.0	
$\delta_0, \eta_0$	-160.6	0.951	-179.1	0.473	-193.8	0.634	-189.8	0.631	-184.9	0.335
	-154.0	0.714	-169.2	0.567	-167.6	0.347	-176.1	0.758	-187.2	0.592
$\delta_1, \eta_1$	-151.9	0.804	-162.7	0.828	-184.5	0.763	-183.5	0.780	-179.2	0.781
	-160.9	0.596	-196.2	0.240	-206.3	0.795	-211.3	0.786	-215.3	0.343
$\delta_2, \eta_2$	-37.6	1.0	-45.9	1.0	-54.0	1.0	-55.0	0.918	-58.4	0.640
	-30.1	1.0	-28.5	0.853	-42.3	0.625	-33.9	0.677	-23.5	0.727
$\delta_3, \eta_3$	141.6	0.560	131.0	0.575	123.8	0.746	123.3	0.733	126.4	0.681
	141.3	0.846	143.2	0.611	129.8	0.488	134.9	0.903	136.9	1.0
$\delta_4, \eta_4$	69.8	0.208	81.5	0.333	76.4	0.503	78.2	0.483	75.7	0.307
	97.5	0.415	113.5	0.830	110.5	0.794	108.4	0.918	105.9	0.718
$\delta_5, \eta_5$	47.6	0.348	57.9	0.437	66.1	0.409	69.3	0.438	65.8	0.512
	25.2	0.432	59.3	0.409	68.7	0.515	64.9	0.312	77.7	0.102
$\delta_6, \eta_6$	1.5	0.727	0.5	0.580	5.6	0.547	12.5	0.500	22.8	0.405
	-6.0	0.557	2.5	0.699	14.5	0.690	6.9	0.415	15.9	0.134
$\delta_7, \eta_7$	8.2	0.896	9.2	0.948	8.7	0.841	12.7	0.807	16.9	0.826
	7.1	0.857	6.3	0.733	3.3	0.742	9.7	0.630	17.5	0.654
$\delta_8, \eta_8$	1.4	0.959	1.8	0.926	1.4	0.882	3.4	0.883	6.0	0.914
	1.5	0.993	2.7	0.993	3.0	0.913	3.9	0.915	5.6	0.873
$\delta_9, \eta_9$	2.4	1.0	1.2	0.992	0.6	0.959	1.9	0.967	4.1	0.986
	2.2	0.963	0.9	0.970	0.6	0.960	1.4	0.945	4.0	0.966

TABLE 5

The ambiguity corresponding to the conjugation of zeros 1 and 2 (cf. fig. 4)

$E_{c.m.s.}$ (MeV)	$\phi_{left}$ (deg.)	$\phi_{right}$ (deg.)	$\sigma_{\text{reac}}$ (mb)	$\sigma'_{\text{reac}}$ (mb)
28.0	8.26	0	473.15	482.87
32.0	4.13	10.27	431.75	431.75
37.0	30.46	14.68	409.14	409.14
40.5	32.59	5.56	391.80	391.80
44.0	13.59	0	381.81	412.87

$\sigma_{\text{reac}}$  is the reaction cross section of the original amplitude at that energy,  $\sigma'_{\text{reac}}$  is the same quantity for the alternative amplitude.

To investigate the dependence of the number of unitary ambiguities on the number of partial waves, we have constructed reduced amplitudes from the first  $L+1$  partial waves of the Tang and Brown amplitude for  $L = 8, 9$  and  $10$  ( $L = 11$  corresponds to the full amplitude). We have then applied the transformations to the reduced amplitudes and counted the numbers of arcs (as in fig. 2) and the number of points (as in fig. 3) for  $L = 8, 9, 10$  and  $11$ . These numbers are displayed in table 2, with their upper bound  $2^L$  (not counting the sign ambiguity). Noteworthy is the steady increase of these numbers as  $L$  increases, although expressed as a fraction of  $2^L$  the number of possibilities actually decreases.

Starting from the Fetscher analysis the ambiguity structure has been determined in a similar fashion at five energies 28, 32, 37, 40.5 and 44 MeV c.m.s. (The breakup channel for seven free nucleons opens at 36 MeV.) The number of partial waves used was ten at all energies. In table 3 we show the number of ambiguity arcs for the ABC transformations and the number of ambiguities for A-transformations (i.e. with the total cross section kept fixed) at these five energies. Again the sign ambiguity  $\delta_l \rightarrow -\delta_l$ , all  $l$ , has not been counted.

One feature of the ambiguous solutions which is quite clear is the fact that the high- $l$  partial waves differ little from the original waves, but that large variations can occur in the lower partial waves.

As remarked in sect. 3, a possible means of reducing or removing the ambiguity is by considering the energy dependence of the position of the zeros of the amplitude in the complex  $z$ -plane. The trajectories of seven out of the ten zeros have been plotted in fig. 4. The remaining three zeros are of a statistical nature and lie much further from the physical region. In order to exhibit the relation between the stable zeros and the minima of the differential cross section we show in fig. 5 the differential cross sections corresponding to the amplitudes of Fetscher and Tang *et al.*

It turned out that we can find the same ABC transforms (conjugating the same subset of zeros at each energy) throughout the whole energy range. In fact we obtained

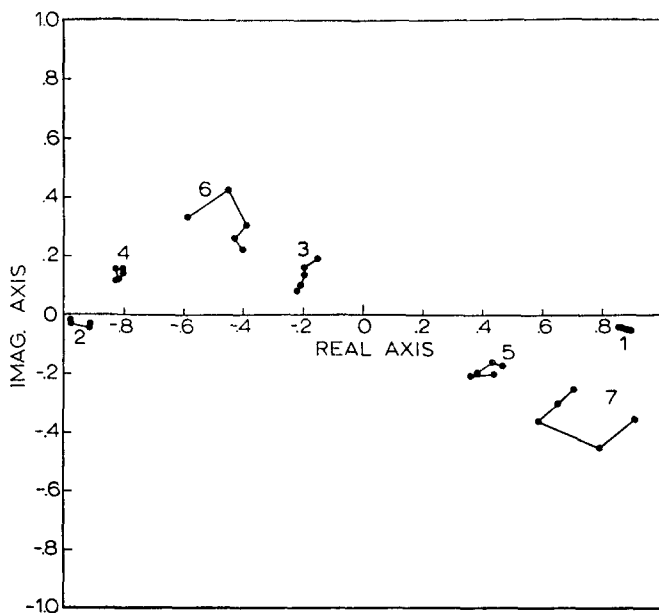


Fig. 4. Positions of the seven most prominent zeros in the complex  $z$ -plane for the  ${}^4\text{He}+{}^3\text{He}$  elastic scattering amplitudes.

two solutions which are acceptable (unitarity) within this energy range; for both cases we have an A-transform at 32, 37 and 40.5 MeV, and an ABC transform at 28 and 44 MeV. The phase shifts and elasticities for one of these ambiguities are shown in table 4. In table 5 we give the corresponding reaction cross sections and the angles  $\phi_{\text{left}} \equiv \phi_{\text{max}} - \arg F(1)$  and  $\phi_{\text{right}} \equiv \arg F(1) - \phi_{\text{min}}$ , with  $\phi_{\text{max}}$  and  $\phi_{\text{min}}$  defined in sect. 2. From the magnitude of  $\phi_{\text{left}}$  and  $\phi_{\text{right}}$  it is clear that our alternative amplitude still possesses a considerable (rotation) freedom of the B-type. Correspondingly there is a large possible variation in  $\sigma_{\text{tot}}$ .

The second process we want to discuss in elastic  $\alpha$ - $\alpha$  scattering. Here we have used two sets of input: the analysis of Darriulat <sup>11</sup>), and the more recent one of Bacher *et al.* <sup>12</sup>). The energy region of interest for our work is from the first inelastic threshold (p-<sup>7</sup>Li) at a c.m.s. energy of 17.4 MeV to about 60 MeV, the highest energy where we have a phase-shift analysis available. Once again only measurements of the differential cross sections have been used in the analysis, so that we can always apply transformation  $B(\phi)$  with  $\phi$  bounded by unitarity. The energies at which a phase-shift analysis is available are closely spaced from 17.4 MeV to 35 MeV. Above that region larger energy steps have been taken.

As we are now considering scattering of identical spinless particles the amplitude will be symmetric in  $z = \cos \theta$ , and will therefore only depend upon  $z^2$ . It is a simple

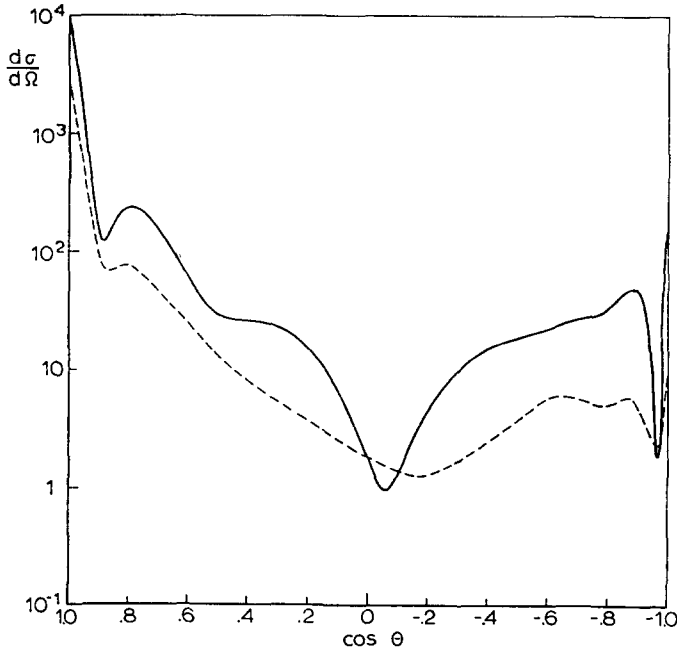


Fig. 5. Plots of  $\sigma(x)$  for  ${}^4\text{He} + {}^3\text{He}$  elastic scattering, calculated from the amplitude of Tang and Brown (44.5 MeV c.m.s., full line) and Fetscher *et al.* (44.0 MeV c.m.s., dotted line).

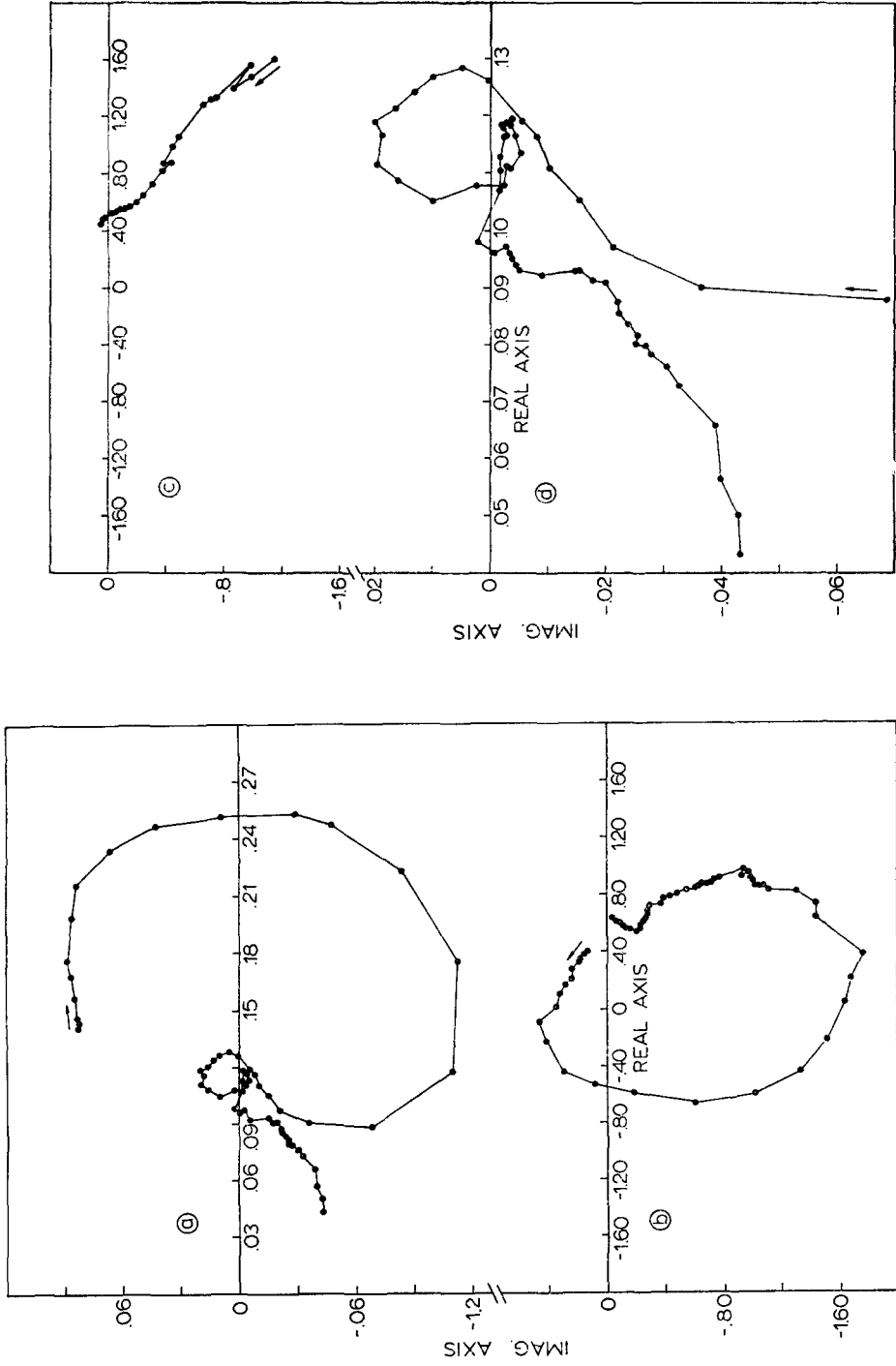


Fig. 6. Zeros in the complex  $z^2$  plane for  ${}^4\text{He} + {}^4\text{He}$  elastic scattering at energies between 17.653 and 34.975 MeV c.m.s.

exercise to formulate the method of sect. 2 in terms of  $z^2$  instead of  $z$ . For convenience we shall therefore consider zeros of the amplitudes in the  $z^2$  plane.

First we discuss in detail the analysis of Bacher *et al.*<sup>12)</sup>. From 17.6 MeV to 27.0 MeV four partial waves have been used in the fit, while at higher energies this number was five. Therefore we have three (at higher energies four) zeros in the  $z^2$  plane. The first three zeros are shown in figs. 6a, b and c. The fourth zero appears only at high energies and does not approach the physical region closely; it always has  $|z|^2 \gg 1$ .

We have applied the transformations ABC to the  $\alpha$ - $\alpha$  scattering amplitudes at these energies. Over a large part of the energy region considered it was possible to conjugate the first zero (fig. 6a) and to obtain again a unitary amplitude. At most of the energies above 24.2 MeV this could be done without applying a B-transformation, and here we fixed the total cross section at the value calculated from the original amplitude. At the energies below 24.2 MeV we had to rotate the new amplitude over a small angle  $\phi$ , which has been chosen in such a way as to give an elastic partial wave  $f_6(\eta_6 = 1)$ . The changes in  $\sigma_{\text{tot}}$  caused by this rotation were always less than 3%. In fig. 6d we have enlarged the relevant part of fig. 6a. It was also possible to conjugate the second zero (fig. 6b), or the third zero (fig. 6c) over a similar range of energies.

We have not made a serious attempt to find a shortest path through all ABC type ambiguities at all energies. This would be a gigantic task, because of the fact that no total cross-section measurements are available to fix the phase of the amplitude, so that at each energy one or several continuous curves of possible alternative solutions are available. We shall therefore only show the sets  $\{\eta_i, \delta_i\}$  for some of the ambiguities we have found choosing the value of the total cross section to be equal (or as close as possible) to the value calculated from Bacher's amplitudes (fig. 7). It is interesting to note that in this set  $\{\eta_i, \delta_i\}$  some new features appear that were not present in the original scattering amplitudes – particularly the clear inelastic effect at about 26 MeV in the S-wave. We readily admit that this particular alternative amplitude is only equivalent to the original amplitude if one neglects the effects of the Coulomb interaction, and that therefore it may not be an acceptable alternative.

The results using the input from the analysis of Darriulat are quite similar to the ones we have displayed starting from the analysis of Bacher *et al.* We therefore will not discuss these results at great length. We only wish to point out that at various (higher) energies Darriulat gave two or three distinct solutions for the phase-shift parameters, each of which corresponded to an acceptable minimum of  $\chi^2$ . Using the techniques developed in the foregoing it is an easy exercise to show that his solutions can in fact be obtained from each other by ABC transformations.

## 5. Conclusion

In this paper we have treated the discrete ambiguity in phase-shift analysis in the inelastic region and we have constructed such ambiguities for  $\alpha$ - $^3\text{He}$  and  $\alpha$ - $\alpha$  scattering amplitudes. As we have remarked, in other work<sup>5)</sup> a different type of ambiguity



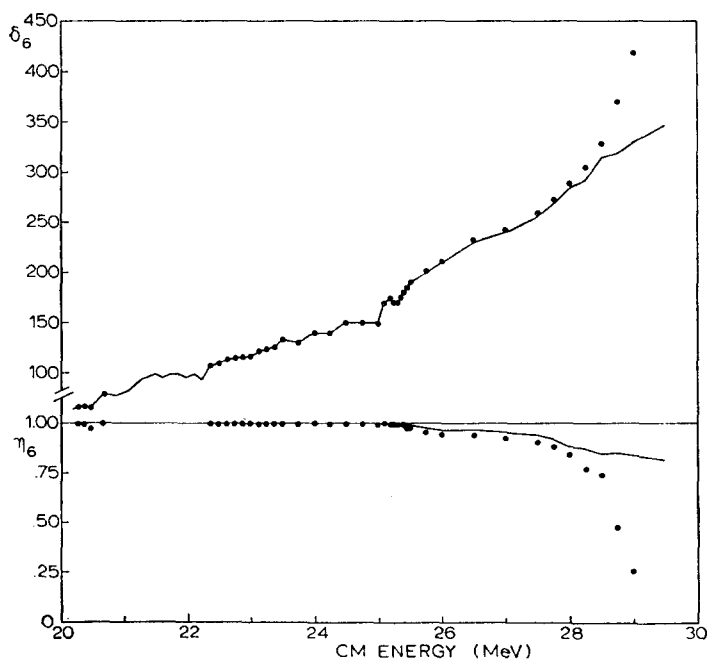
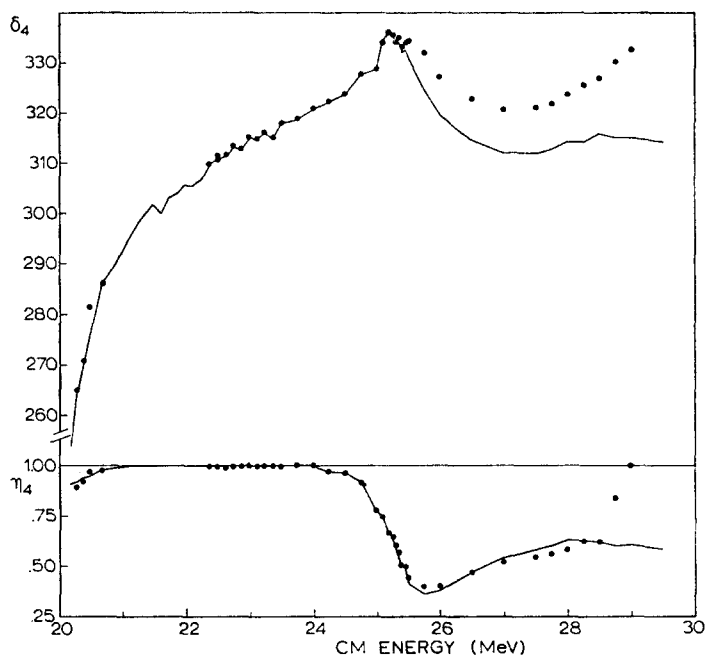


Fig. 7a.

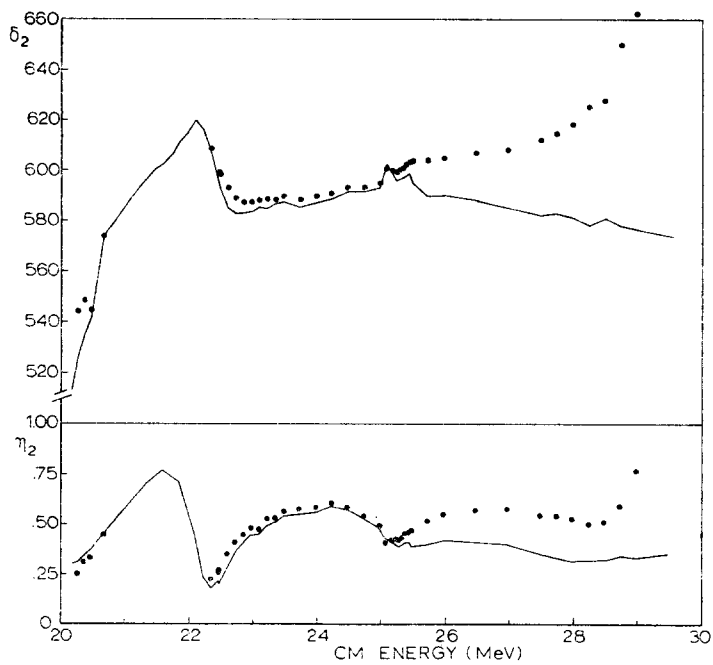
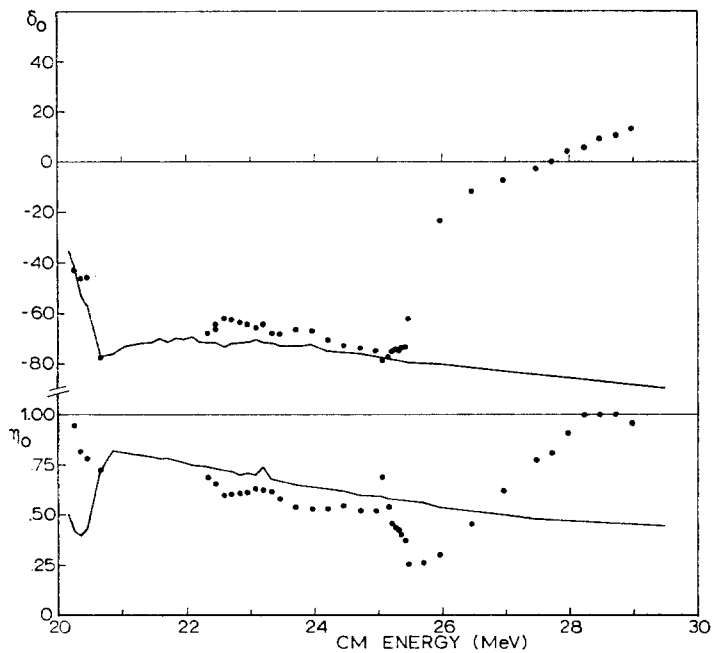


Fig. 7b.

Fig. 7. Two alternative sets of phase-shift parameters  $\eta$  and  $\delta$  as a function of the energy for  $\alpha$ - $\alpha$  scattering.

has been investigated. This is the so-called continuum ambiguity, which has been explored for  $\alpha$ - $\alpha$  scattering in the same energy range as in the present paper. It has been shown that the continuum ambiguity obtained with the Darriulat amplitude as a starting point is quite considerable. Each of the ambiguities produced by the transformations of sect. 2 could also be used as starting points for the construction of continuous ambiguities. It appears likely that the full extent of the ambiguity would then be increased.

The main purpose of this and related papers has been to construct as many ambiguous amplitudes as possible, and to point out in this way the many hazards of a conventional phase-shift analysis. The problem of finding the correct physical scattering amplitude is left unresolved. One can only hope to solve this problem by introducing new constraints, in the form of more experimental data or of theoretical models, into the phase-shift problem. To conclude, we briefly comment on some of these constraints.

(i) In the preceding sections we have shown that knowledge of the total cross section fixes the phase of the forward amplitude up to a sign.

(ii) If one can measure the angular distributions of the inelastic processes one can construct a new kind of phase-shift problem by using the full multichannel unitarity relation. If all cross sections  $\sigma_{cc'}(\theta)$ , where  $c$  and  $c'$  label all open channels, are measured the continuum ambiguity will be resolved<sup>5)</sup>. Because all inelasticity parameters are determined by the multi-channel analysis only accidental discrete ambiguities remain. These would be of the same character as the Crichton ambiguity in the elastic region.

(iii) Dynamical information can be used either to select solutions from among the ambiguities, or can be injected into the phase-shift problem at the stage of the  $\chi^2$  fitting. If, for instance, the behaviour of certain partial waves can be predicted, one can use this as a selection criterion. Such predictions could come from theoretical models or measurements of other processes. Sometimes it is possible to parametrize the amplitude in such a way that features predicted by a model are built in. Of course the resulting amplitude is then only as believable as the model.

(iv) As we remarked in sect. 1, for charged particles the phase-shift ambiguities can in principle be removed by using the singular properties of the Coulomb amplitude  $F_C$ .

(v) The requirement that a physical amplitude behaves smoothly with energy is a strong constraint on the choice of solutions at each energy. In sect. 4 we have shown in practice that this constraint is indeed powerful, but the examples of sect. 3 (fig. 1) as well as the result in sect. 4 (fig. 6) show that it is not always possible to determine the amplitude uniquely.

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